

Why Use a Hamilton Approach in QCD?

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Abstract

We discuss QCD in the Hamiltonian frame work. We treat finite density QCD in the strong coupling regime. We present a parton-model inspired regularisation scheme to treat the spectrum (θ -angles) and distribution functions in QED_{1+1} . We suggest a Monte Carlo method to construct low-dimensionasl effective Hamiltonians. Finally, we discuss improvement in Hamiltonian QCD .

1. Introduction

There are presently the following problems in QCD : How to compute non-perturbatively (on the lattice): (a) QCD at non-zero chemical potential - finite quark density. (b) Hadron structure functions, at small x_B and small Q . (c) S -matrix, scattering cross sections, phase shifts, (d) Decay amplitudes. Here, we want to discuss recent progress in the Hamiltonian formulation, QCD at non-zero chemical potential in the strong coupling regime, distribution function in QED_{1+1} , the Monte Carlo Hamiltonian, and improvement of the continuum behavior of the Hamiltonian.

2. Finite density QCD

According to the big bang model of cosmology creation of quarks and gluons occured at about 10^{-35} sec, quark-gluon plasma (QGP) was formed at about 10^{-6} sec, then there was a phase transition and the formation of protons and neutrons began at 10^{-4} sec. Today the universe is in a low temperature and low density phase: quarks are confined. Quark-gluon plasma may exist in cores of very dense stars like neutron stars. Experimental search for QGP is carried out at the Relativistic Heavy Ion Collider (RHIC) at BNL and at the Large Hadron Collider (LHC) at CERN. Theorists search for a precise determination of phase structure of QCD at finite temperature T and finite chemical potential μ (finite density). Shuryak [1] has predicted that QCD

at high temperature and density displays beyond the QGP phase transition a much richer phase structure, due to strong instanton effects.

Here we suggest to consider finite temperature and finite density QCD in the Hamiltonian formulation. Consider the grand canonical partition function at temperature T and chemical potential μ

$$Z = \text{Tr} e^{-\beta(H - \mu N)}, \quad N = \int d^3x \psi^\dagger(x) \psi(x), \quad \beta = (k_B T)^{-1}. \quad (1)$$

H is the Hamiltonian and N is the fermion (quark) particle number operator, No complex action problem occurs! Let us consider as example the energy density for free quarks

$$\epsilon = \frac{1}{V} \frac{1}{Z} \text{Tr} H e^{-\beta(H - \mu N)} = -\frac{1}{V} \frac{\partial \ln Z}{\partial \beta} \Big|_{\mu, \beta}. \quad (2)$$

Taking the zero-temperature limit ($T \rightarrow 0$) and the chiral limit ($m \rightarrow 0$) the energy density (with the contribution $\mu = 0$ subtracted) becomes $\epsilon_{sub} = \frac{\mu^2}{4\pi^2}$. The same result is obtained by starting from a lattice Hamiltonian with Wilson fermions.

Strong coupling QCD at non-zero chemical potential.

Lattice QCD at $\mu = 0$ confines quarks and spontaneously breaks chiral symmetry. Note: The strong coupling regime $1/g^2 \ll 1$ is *not* compatible with continuum limit $a \rightarrow 0$. The goal of the following calculation is to get a better understanding of mechanism of chiral phase transition. We make an ansatz for the lattice Hamiltonian in the strong coupling regime following Ref.[2]. Then the following results are obtained [3]. The vacuum energy becomes

$$E_\Omega = 2N_f N_c N_s \left[(m_{dyn}^{(0)} - \mu) \Theta(\mu - m_{dyn}^{(0)}) - m_{dyn}^{(0)} - \mu \right] \\ m_{dyn}^{(0)} = \frac{d}{ag^2 C_N}, \quad C_N = (N_c^2 - 1)/(2N_c), \quad (3)$$

where $m_{dyn}^{(0)}$ is the dynamical quark mass at $\mu = 0$. In order to compute the chiral condensate and the critical chemical potential one can use the Feynman-Hellmann theorem. The chiral condensate at $\mu = 0$ becomes

$$\langle \bar{\psi} \psi \rangle^{(0)} = -2N_c [1 - 4d/(g^2 C_N^2)], \quad (4)$$

and the chiral condensate at $\mu \neq 0$ becomes

$$\langle \bar{\psi} \psi \rangle = \langle \bar{\psi} \psi \rangle^{(0)} \left[1 - \Theta(\mu - m_{dyn}^{(0)}) \right]. \quad (5)$$

As a result one has

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &\neq 0 \text{ for } \mu < m_{dyn}^{(0)} : \text{chiral symmetry broken,} \\ \langle \bar{\psi}\psi \rangle &= 0 \text{ for } \mu > m_{dyn}^{(0)} : \text{chiral symmetry restored.} \end{aligned} \quad (6)$$

Thus we make the prediction of a first order chiral phase transition. The critical chemical potential is $\mu_{crit} = m_{dyn}^{(0)}$. Our result for μ_{crit} is in agreement with the result obtained by Le Yaouanac et al.[4] However, Le Yaouanac et al. find a phase transition of 2nd order. There have been predictions of the nucleon mass [4] $M_{nucl}^{(0)} \approx 3m_{dyn}^{(0)}$. Thus our result is $\mu_{crit} \approx M_{nucl}^{(0)}/3$, being consistent with other theoretical predictions [4]. We have moreover computed the quark number density, the corresponding susceptibility and some thermal masses [3].

3. Hamiltonian in a fast moving frame

At hand of the massive Schwinger model, QED_{1+1} , we want to discuss the following topics: (i) A physically motivated many-body method to solve a Hamiltonian many-body problem and compute structure functions. (ii) The treatment of the θ -angle in Hamiltonian gauge theory. The motivation for the Hamiltonian many-body method comes from the parton model of deep inelastic lepton-hadron scattering. Experimentally, the Breit frame is a convenient choice for the interpretation of structure functions. We suggest to use a fast moving frame also in a computational study [5]. The Hamiltonian in axial gauge $A^3 = 0$ is given by

$$H = \int_{-L}^L dx^3 (\bar{\psi}\gamma^3 i\partial_3\psi + m\bar{\psi}\psi) + \frac{g^2}{2} \int_{-L}^L dx^3 (\psi^\dagger\psi) \frac{1}{-\partial_3^2} (\psi^\dagger\psi). \quad (7)$$

We start from a space-time lattice of spacing a and length $[-L, L]$. Then we go over to a momentum lattice with cut-off $\Lambda = \pi/a$ and resolution $\Delta p = \pi/L$. We use the assumption (parton model): Left moving particles are not dynamically important, when *physical* particles are considered from a reference frame with velocity $v = P/E \approx c$. On the momentum lattice the following bounds hold: $0 \leq p^0, p^3 \leq P$. A fast moving particle is Lorentz contracted. It fits into a small lattice volume (compared to rest frame). A small lattice size implies a small number of virtual particle pairs created from vacuum: Number of vacuum pairs \propto vacuum density \times lattice size. For the scaling window holds: $a \leq \text{Compton wavelength} \leq L$ and $a \leq \xi a \leq L$, where ξ is the correlation length in dimensionless units and $M = \frac{1}{\xi a}$ is the physical mass of particle in the ground state. In the strong relativistic regime holds $M \ll P < \Lambda$. In the limit $a \rightarrow 0$ the scaling window is replaced by: $1/P \leq 1/M \leq L$. In Ref.[5] we have computed the mass spectrum: Vector boson and scalar

boson, and compared it with predictions of chiral perturbation theory. We also determined the dependence on the θ angle, defined by a global gauge transformation $\Psi[A] \rightarrow e^{in\theta} \Psi[A]$, $n = 0, \pm 1, \pm 2, \dots$. The θ -action S and topological charge q are related by $S[A, \theta] = S[A] - \theta q[A]$. In Ref. [5] we have also studied the massive Schwinger model and its dependence on the θ -angle in the chiral limit. For the massless Schwinger model the chiral condensate is analytically known,

$$\frac{\langle \bar{\psi}\psi \rangle}{g} = \frac{e^\gamma}{2\pi\sqrt{\pi}} \cos(\theta) \approx 0.1599 \cos(\theta). \quad (8)$$

Using the Feynman-Hellmann theorem to relate the vector mass and the chiral condensate, we determine the slope from numerical data and obtain $\frac{\langle \bar{\psi}\psi \rangle}{g} \approx 0.16 \cos(\theta)$, which is close to the analytical result.

The Hamiltonian approach also allows to compute wave functions, giving information on the parton structure.

$$f(x_B) = \langle \Psi_V(P) | \frac{1}{2} [b_k^\dagger b_k + d_k^\dagger d_k] | \Psi_V(P) \rangle, \quad (9)$$

where $x_B = \frac{k}{P}$ is the fraction of momentum of vector boson carried by parton. One observes that a large lattice ($N=200$) is needed to resolve distribution functions precisely (see Ref.[5]).

4. Stochastic Hamiltonian

Lagrangian lattice field theory and its non-perturbative numerical solution by Monte Carlo methods is very successful. The typical number of configurations (depending on the statistical error) is in the order of $\sim 10^2$. In Ref.[6,7], we have proposed to construct an effective Hamiltonian, by treating the many-body problem by applying as far as possible Monte Carlo with importance sampling (e.g. Metropolis algorithm) in analogy to Lagrangian lattice simulations. Let us consider as example 1-D Q.M. We compute transition matrix elements in imaginary time

$$M_{ij}(T) = \langle x_i | e^{-HT/\hbar} | x_j \rangle, \quad i, j \in 1, \dots, N, \quad (10)$$

where x_i are discrete position eigen states. We define an effective Hamiltonian via eigenfunctions and eigenvalues:

$$M(T) = U^\dagger D(T) U, \quad U_{ik}^\dagger = \langle e_i | E_k^{eff} \rangle, \quad D_i(T) = e^{-E_i^{eff} T/\hbar}. \quad (11)$$

We compute matrix elements using Monte Carlo with importance sampling

Fig. 1. Harmonic oscillator. Specific heat $C(\beta)$ from Monte Carlo Hamiltonian, using stochastic basis.

Fig. 2. QCD_{1+1} . Fermionic improvement. $\chi = -\langle \bar{\psi}\psi \rangle_{sub}/(g_{lat}N_c)$ versus $1/g_{lat}^2$ for $N_C = 3$ with Wilson fermions. Crosses: $r = 0.1$, Diamonds: $r = 1$.

Fig. 3. QCD_{1+1} . Fermionic improvement. $\chi = -\langle \bar{\psi}\psi \rangle_{sub}/(g_{lat}N_c)$ versus $1/g_{lat}^2$ for $N_C = 3$ with improved Wilson fermions. Crosses: $r = 0.1$, Diamonds: $r = 1$.

(e.g. Metropolis algorithm). Splitting the action $S = S_0 + S_V$, we treat $O = \exp[-S_V/\hbar]$ as observable and compute the matrix element by Monte Carlo

$$M_{ij}(T) = M_{ij}^{(0)}(T) \times \frac{\int [dx] \exp[-S_V[x]/\hbar] \exp[-S_0[x]/\hbar] |x_i, T\rangle_{x_j, 0}}{\int [dx] \exp[-S_0[x]/\hbar] |x_i, T\rangle_{x_j, 0}}. \quad (12)$$

The construction of a stochastic basis has been proposed in Ref.[7]. The result of a numerical study is shown in Fig.[1].

5. Improvement in Hamiltonian QCD

Szymanzik [8] suggested improvement program for lattice field theory: Construction of "improved" action, such that error due to finite lattice spacing a becomes smaller. Progress in improvement for lattice pure gauge theory was introduced by Lepage et al. [9], suggesting tadpole improvement. We have suggested how to construct an improved Hamiltonian on the lattice. For pure gauge theory the construction of an improved Hamiltonian for $SU(3)$ gauge theory via the transfer matrix has been reported by Luo et al. in Ref.[10]. Improvement of the fermionic sector of a lattice Hamiltonian has been carried out for QCD_{1+1} by Luo et al. in Ref.[11]. A numerical study was carried out by Jiang et al. [12]. An example is shown in Fig.[2,3], showing that improvement reduces the dependence on Wilson's r -parameter.

6. Conclusion

Hamiltonian formulation of QCD is attractive, for the following reasons: (a) There is no sign or complex action problem! (b) It is suitable for the computation of wave functions and structure functions. (c) It is suitable for the computation of the S-matrix. The problem is: How to solve the Hamiltonian many-body system? (d) Proposal: Construct an effective Hamiltonian via Monte Carlo. (e) Improvement program: Also possible in Hamiltonian formulation.

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